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LETTER TO THE EDITOR

# Path integration of a three body problem

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**Abstract.** Path integration for a one dimensional three body problem is discussed. This involves essentially the path integration of a Lagrangian in plane polar coordinates with a potential energy term depending on both the radial and angular coordinates.

In a recent paper (Khandekar and Lawande 1972), the authors have presented a path integral for a one dimensional three body problem. This problem considered three equal mass particles in one dimension, with equal strength harmonic forces between every two particles, and an additional interaction which varies as the inverse square of the distance between any one pair of particles. In this note, we discuss the path integration of the more general three body problem with harmonic and inverse square potentials between every pair of particles. This case is interesting because it involves a nontrivial extension of the one considered earlier.

The problem is characterized by the Lagrangian

$$L = \tfrac{1}{2}(\dot{x}_1{}^2+\dot{x}_2{}^2+\dot{x}_3{}^2)-\tfrac{1}{4}\omega^2\{(x_1-x_2)^2+(x_2-x_3)^2+(x_3-x_1)^2\}$$
$$-g\{(x_1-x_2)^{-2}+(x_2-x_3)^{-2}+(x_3-x_1)^{-2}\} \qquad (1)$$

where $x_1$, $x_2$, $x_3$ are the coordinates of the three particles, $\omega$ the angular frequency arising from the strength of the harmonic potentials, and $g$ the strength of the inverse square potentials. We assume $g > -\frac{1}{4}$ to avoid the two body collapse and use units such that $\hbar = m = 1$, where $m$ is the mass of the particles. Introducing the centre of mass (CM), Jacobi coordinates and subsequently polar coordinates $(r, \phi)$, the Lagrangian of equation (1) is transformed as

$$L = \frac{3}{2}\dot{R}^2+\frac{1}{2}(\dot{r}^2+r^2\dot{\phi}^2)-\frac{3}{4}\omega^2r^2-\frac{9g}{2r^2\sin^2 3\phi}. \qquad (2)$$

The centre of mass term $\frac{3}{2}\dot{R}^2$ in the Lagrangian yields a free particle propagator and is of no further interest. Our aim is to carry out the path integration of a 'two dimensional classical one body problem' in polar coordinates involving a potential depending on both $r$ and $\phi$, and described by the Lagrangian

$$\mathscr{L} = \frac{1}{2}(\dot{r}^2+r^2\dot{\phi}^2)-\frac{3}{4}\omega^2r^2-\frac{9g}{2r^2\sin^2 3\phi}. \qquad (3)$$

This case differs from the one considered in the earlier paper (Khandekar and Lawande 1972) by the presence of $\sin^2 3\phi$ instead of $\sin^2 \phi$ in the 'centrifugal potential' term. To perform the path integration, we first consider the action $S(\boldsymbol{r}_j, t_j; \boldsymbol{r}_{j-1}, t_{j-1})$ ($\boldsymbol{r}_j = \boldsymbol{r}(t_j)$, $\boldsymbol{r}_{j-1} = \boldsymbol{r}(t_{j-1})$) over an infinitesimally small time interval $t_j - t_{j-1} = \epsilon$:

$$S(\boldsymbol{r}_j, t_j; \boldsymbol{r}_{j-1}, t_{j-1}) = \frac{1}{2\epsilon}\{r_j^2 + r_{j-1}^2 - 2r_j r_{j-1}\cos(\phi_j - \phi_{j-1})\} - \frac{3}{4}\omega^2 r_j^2 - \frac{9g}{2r_j r_{j-1}\sin 3\phi_j \sin 3\phi_{j-1}}. \quad (4)$$

Next, we have to express $\cos(\phi_j - \phi_{j-1})$ in terms of $\cos 3(\phi_j - \phi_{j-1})$ correctly. Writing $\psi = \phi_j - \phi_{j-1}$ for short, it is easy to obtain

$$\cos\psi \simeq \frac{1}{9}(8 + \cos 3\psi - 3\psi^4) + \mathrm{O}(\psi^6). \quad (5)$$

As pointed out by Edwards and Gulyaev (1964), it is necessary to retain the fourth order terms in the angular changes $\psi$ in order to obtain the correct finite time propagator. We note further that only terms of $\mathrm{O}(\epsilon)$ need be retained in the action, and use the following easily verified intermediate results:

$$\psi^4 \exp\left(\frac{u}{\epsilon}\cos\psi\right) \simeq \frac{3\epsilon^2}{u^2}\exp\left(\frac{u}{\epsilon}\cos\psi\right) + \mathrm{O}(\epsilon^3) \quad (6)$$

$$\exp\frac{u}{\epsilon}(\cos 3\psi - 3\psi^4) \simeq \exp\left(\frac{u}{\epsilon}\cos 3\psi - \frac{\epsilon}{9u}\right) + \mathrm{O}(\epsilon^2) \quad (7)$$

$$I_a\left(\frac{u}{\epsilon}\right) \simeq \left(\frac{\epsilon}{2\pi u}\right)^{1/2} \exp\left\{\frac{u}{\epsilon} - \frac{1}{2}\left(a^2 - \frac{1}{4}\right)\frac{\epsilon}{u} + \mathrm{O}(\epsilon^2)\right\} \quad (8)$$

$$I_b\left(\frac{u}{\epsilon}\right)\exp\left(\frac{8u}{\epsilon} - \frac{\epsilon}{9u}\right) \sim 3I_{3b}\left(\frac{9u}{\epsilon}\right) + \mathrm{O}(\epsilon^2) \quad (9)$$

and the expansion formula (Erdélyi 1953, p 102)

$$(\sin\alpha\sin\beta)^{1/2-\lambda} I_{\lambda-1/2}(u\sin\alpha\sin\beta)\exp(u\cos\alpha\cos\beta) = 2^{2\lambda}(2\pi u)^{-1/2}(\Gamma(\lambda))^2 \sum_{l=0}^{\infty} \frac{l!(\lambda+l)}{\Gamma(2\lambda+l)} I_{l+\lambda}(u) C_l^\lambda(\cos\alpha) C_l^\lambda(\cos\beta). \quad (10)$$

Thus, setting

$$u = \frac{r_j r_{j-1}}{9\mathrm{i}} \qquad a = \tfrac{1}{2}(1+4g)^{1/2} \qquad \lambda = a + \tfrac{1}{2} \quad (11)$$

and

$$\alpha = 3\phi_j \qquad \beta = 3\phi_{j-1} \qquad l = l_j \qquad b = l_j + a + \tfrac{1}{2} \quad (12)$$

and using equations (5–12), we obtain as before (Khandekar and Lawande 1972) the propagator for infinitesimal time interval

$$K(\boldsymbol{r}_j, t_j; \boldsymbol{r}_{j-1}, t_{j-1}) = \left(\frac{1}{2\pi\mathrm{i}\epsilon}\right)\exp\{\mathrm{i}S(\boldsymbol{r}_j, t_j; \boldsymbol{r}_{j-1}, t_{j-1})\}$$
$$= \left(\frac{1}{2\pi\mathrm{i}\epsilon}\right) N_{l_j}^2 (\sin 3\phi_j \sin 3\phi_{j-1})^{a+1/2} C_{l_j}^{a+1/2}(\cos 3\phi_j) \times C_{l_j}^{a+1/2}(\cos 3\phi_{j-1}) R_{l_j}(r_j, r_{j-1}) \quad (13)$$

where $N_{l_j}$ is the normalization factor of the Gegenbauer polynomial $C_{l_j}^{a+1/2}(\cos 3\phi_j)$ (Erdélyi 1953, p 174) and

$$R_{l_j}(r_j, r_{j-1}) = 2\pi \exp\left(\frac{i}{2\epsilon}(r_j^2 + r_{j-1}^2) - \frac{3i\epsilon\omega^2 r_j^2}{4}\right) I_{3(l_j+a+1/2)}\left(\frac{r_j r_{j-1}}{i\epsilon}\right). \tag{14}$$

The finite time propagator may then be obtained by successive iteration. After performing the intermediate angular integrations (over the range $0 \leqslant \phi \leqslant \pi/3$), we arrive at

$$K(r'', \phi''; r', \phi', t) = \sum_{l=0}^{\infty} K_l(r'', r', t) N_l^2 (\sin 3\phi'' \sin 3\phi')^{a+1/2} \times C_l^{a+1/2}(\cos 3\phi'') C_l^{a+1/2}(\cos 3\phi') \tag{15}$$

where

$$K_l(r'', r', t) = \lim_{N\to\infty} \left(\frac{1}{2\pi i\epsilon}\right)^N \int \prod_{j=1}^{N} R_{l_j}(r_j, r_{j-1}) \prod_{j=1}^{N-1} r_j \mathrm{d}r_j \tag{16}$$

is the radial propagator of the $l$ wave. Since $K_l(r'', r', t)$ involves only the harmonic potential, it is possible to evaluate it in a closed form as before (Khandekar and Lawande 1972). Thus

$$K_l(r'', r', t) = \left(\frac{\Omega}{i \sin \Omega t}\right) \exp\left(\frac{i\Omega}{2}(r'^2 + r''^2) \cot \Omega t\right) \times I_{3(l+a+1/2)}\left(\frac{\Omega r'' r'}{i \sin \Omega t}\right) \qquad (\Omega = \sqrt{\tfrac{3}{2}}\,\omega). \tag{17}$$

Finally, the expansion of the propagator in terms of the eigenfunctions of the corresponding Schrödinger equation determined by Calogero (1969), may be obtained as in the earlier paper.

The main points of interest in this work are that (i) it involves the path integration of the Lagrangian of equation (3) which contains a potential term which depends on both $r$ and $\phi$, (ii) the special form of the 'centrifugal' potential term, namely $g/2r^2 \sin^2 3\phi$, yields a natural separation of the propagator into radial and angular parts; physically this is related to the conservation of the classical quantity $(p_\phi^2 + g/\sin^2 3\phi)$ where $p_\phi$ is the angular momentum, and (iii) it requires the use of equation (5) which once more brings out the rule about path integration in polar coordinates first emphasized by Edwards and Gulyaev (1964).

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